On an Almost Complex Manifold Approach to Elementary Particles

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An almost complex manifold description of elementary particles is proposed which links the approaches given independently by J. Ławrynowicz and L. Wojtczak, and by C. von Westenholz. This description leads to relations between the curvature form of an almost complex manifold, which accounts for the symmetry classification schemes within the frame of principal fibre bundles, and a curved Minkowski space-time via induced smooth mappings characterizing nuclear reactions of type $\mathrm{N} + \pi \rightleftarrows \mathrm{N}$, where N is some nucleon and π the virtual π-meson of this reaction. Both approaches follow the same main idea of D. A. Wheeler developed in a different way by A. D. Sakharov.

Introduction

Ławrynowicz and Wojtczak [1] proposed a description of the properties of elementary particles by a suitable choice of three fibre bundles [2]. One of these bundles has as its fibre space a pseudo-riemannian [1] manifold $\mathbb{M}$ — the space of observations (a curved Minkowski space-time), two other bundles have as their fibre space general Riemannian manifolds $\mathbb{N}_\mathrm{e}$ and $\mathbb{N}_\mathrm{n}$ — the spaces of the particle, connected with the external electromagnetic and nuclear fields, respectively. Von Westenholz [3] found independently that the underlying structure of interacting fields can be expressed in terms of de Rham cohomology, i.e. the properties of elementary particles can be explained by means of pairings (ω^p, c_p), where ω^p stands for a C^1-differentiable p-form, c_p for a p-chain, $p = 0, 1, 2, 3, 4$, and the geometry is derived from the principal fibre bundle $P(M^4, G)$ with a structural group G over a curved Minkowski space-time M^4. Within the framework of his theory pairings (ω^p, c_p) give a description of the interacting fields without any Lagrangian formalism. Such an approach allows us to avoid the main difficulties inherent in conventional field theories [4].

The present authors assume that $M^4 = \mathbb{M}$ and establish the relations between the curvature form Ω^2 of some principal fibre bundle over an almost complex manifold $\mathbb{L}$ derived from $\mathbb{N}_\mathrm{e}$ and $\mathbb{N}_\mathrm{n}$, and the curved space-time $\mathbb{M}$ via some induced C^2-smooth mappings $v_\mathrm{e}\colon \mathbb{N}_\mathrm{e} \to \mathbb{M}$ and $v_\mathrm{n}\colon \mathbb{N}_\mathrm{n} \to \mathbb{M}$. The form Ω^2 stands for the meson field M and the baryon field B, namely

$$\Omega^2 = M + i\,B\,, \tag{1}$$

where i denotes the imaginary unit: $i^2 = -1$, and the matrices M and B will be commented on in the next section.

From the point of view of our present purpose the choice of M and B is irrelevant, since we are going to express various physical quantities in terms of M and B. This means that our further considerations fit to several motivated concepts of a choice of them, and it is perhaps better to keep at this stage their arbitrariness as far as possible.

Namely, the form Ω^2 accounts for symmetry classification schemes within the frame of principal fibre bundles, while the mappings v_e and v_n characterize nuclear reactions of the type $\mathrm{N} + \pi \rightleftarrows \mathrm{N}$, where N is some nucleon and π the virtual π-meson of this reaction. The topological and metrical properties of the curved space-time $\mathbb{M}$ manifest themselves as electric charge, magnetic dipole, nuclear charge, nuclear dipole, and mass. Thus the relations in question are the key relations for a classification scheme of elementary particles and link both the concepts given in Ref. [1] and Ref. [3], supporting the hypothesis that the postulates in Ref. [1] correspond to the theorems in Ref. [3] and vice versa.

1. The Curvature Form Related to the Meson and Baryon Fields

Within the approach proposed in Ref. [1] there is still some freedom in the choice of the pseudo-riemannian metric g of $\mathbb{M}$. If we accept the point

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of view suggested by Sakharov[5] (based on the ideas of Wheeler[6]), we may set, in local coordinates,

$$g_{jk} = \tfrac{1}{2} \sum_{l} \left(\frac{\partial \psi_l}{\partial x^j} \frac{\partial \overline{\psi}_l}{\partial x^k} + \frac{\partial \psi_l}{\partial x^k} \frac{\partial \overline{\psi}_l}{\partial x^j} \right) \tag{2}$$

where ψ is the field four-vector corresponding to the system of particles considered. Thus we may interpret the results of this paper as relations between the curvature form Ω^2 and the pseudo-riemannian metric g. Therefore we arrive, at least theoretically, at a consistent description of the symmetries connected with Ω^2, i.e. with the own spaces of particles.

Since practically it is quite cumbersome to write Ω^2 explicitly, one may try the choice

$$M = \begin{bmatrix} 2^{-\frac{1}{2}}\pi^0 + 6^{-\frac{1}{2}}\eta & \pi^+ & \mathrm{K}^+ \\ \pi^- & -2^{-\frac{1}{2}}\pi^0 + 6^{-\frac{1}{2}}\eta & \mathrm{K}^0 \\ \mathrm{K}^- & \overline{\mathrm{K}}{}^0 & -2\cdot 6^{-\frac{1}{2}}\eta \end{bmatrix}, \tag{3}$$

$$B = \begin{bmatrix} 6^{-\frac{1}{2}}\Lambda + 2^{-\frac{1}{2}}\Sigma^0 & \Sigma^+ & \mathrm{p} \\ \Sigma^- & 6^{-\frac{1}{2}}\Lambda - 2^{-\frac{1}{2}}\Sigma^0 & \mathrm{n} \\ \Xi^- & \Xi^0 & -2\cdot 6^{-\frac{1}{2}}\Lambda \end{bmatrix}, \tag{4}$$

which is natural in the frames of von Westenholz's approach[3]. The elements of matrices M and B are C^1-differentiable two-forms corresponding to the $SU(3)$ classification table, where the Lie algebra of $SU(3)$ is generated, as usually, by the elements $A_k{}^j$; $j, k = 1, 2, 3$ (cf. Ref.[7]).

Because $\mathbb{L}$ is a product almost complex manifold, it would be convenient to have the elements of M and B real-valued. If, however, we assumed a Lagrangian field theory, this is impossible. Indeed, from the physical point of view the complex character of the matrices M and B is connected with the appearance of particles and antiparticles which could be of an opposite sign of the charge. In this case the particle would be described by the matrix M or B while the antiparticle by their complex conjugates.

If we do not assume a Langrangian field theory which is the case when giving quantization in terms of flux quantization[1,3] so that we avoid the usual second quantization, the elements of M and B could be regarded as real-valued.

After discovering the mesons $\psi_1(3095)$[8] and $\psi_2(3684)$[9] it is, of course, natural to replace the matrices M and B given by (3) and (4) by some higher-dimensional analogues

$$M = M', \qquad B = B' \tag{5}$$

which will be specified in a subsequent paper.

2. A Description of Elementary Particles in Terms of Almost Complex Structures

The above programme motivates to set up ab ovo foundations of the physics of elementary particles in terms of almost complex manifolds endowed with an hermitian metric. Let us denote by top $\mathbb{N}_e$ and top $\mathbb{N}_n$ the paracompact[10] topological spaces corresponding to $\mathbb{N}_e$ and $\mathbb{N}_n$, and by g_e and g_n their general[1] Riemannian metrics. Let top $\mathbb{L}$ = top $\mathbb{N}_e$ $\times$ top $\mathbb{N}_n$. When introducing an hermitian structure on $\tilde{\mathbb{L}}$ = (top $\mathbb{L}$, atl $\mathbb{L}$), a differentiable manifold with the maximal atlas atl $\mathbb{L}$, we have to suppose that $\tilde{\mathbb{L}}$ admits an almost complex structure J. For any $z = (x, y)$, where x and y are points of $\mathbb{N}_e$ and $\mathbb{N}_n$, respectively, we denote by $\mathrm{pr}_z{}^e$ the mapping induced by the projection $z \mapsto x$ between the corresponding tangent spaces: $\mathrm{pr}_z{}^e \colon T_z\mathbb{L} \to T_x\mathbb{N}_e$ and, analogously, we introduce $\mathrm{pr}_z{}^n \colon T_z\mathbb{L} \to T_y\mathbb{N}_n$. Formula (3.1) in Ref. [1] suggests to concentrate first at investigating the hermitian structures given by the formula

$$h(\mathfrak{v}, \mathfrak{w}) = \hat{g}(\mathfrak{v}, \mathfrak{w}) - i\,\hat{g}(J\mathfrak{v}, \mathfrak{w}) + i\,\hat{g}(\mathfrak{v}, J\mathfrak{w}) + \hat{g}(J\mathfrak{v}, J\mathfrak{w}), \tag{6}$$

where $\mathfrak{v}$, $\mathfrak{w}$ belong to $T_z\mathbb{L}$, and

$$\hat{g}(\mathfrak{v}, \mathfrak{w}) = g_e(\mathrm{pr}_z{}^e\mathfrak{v}, \mathrm{pr}_z{}^e\mathfrak{w}) + g_n(\mathrm{pr}_z{}^n\mathfrak{v}, \mathrm{pr}_z{}^n\mathfrak{w}).$$

The required hermitian manifold is defined by

$$\mathbb{L} = (\tilde{\mathbb{L}}, J, h) = (\mathrm{top}\,\mathbb{L}, \mathrm{atl}\,\mathbb{L}, J, h). \tag{7}$$

We assume now that the structure group G in $P(\mathbb{M}, G)$ is $SU(2) \otimes G^*$ in the case given by relations (3) and (4). We also consider the almost complex principal fibre bundle $P(\mathbb{L}, G)$ with $G = SU(2) \otimes G^*$ as well as the corresponding principal fibre bundles $P(\mathbb{N}_e, G')$ and $P(\mathbb{N}_n, G')$ with $G' = SU(3)$. Similarly we assume that the structure group G in $P(\mathbb{M}, G)$ is $SU(3) \otimes G^*$ in the case given by relations (5). We also consider the almost complex principal fibre bundle $P(\mathbb{L}, G)$ with $SU(3) \otimes G^*$ as well as the corresponding principal fibre bundles $P(\mathbb{N}_e, G')$ and $P(\mathbb{N}_n, G')$ with $G' = SU(4)$. The choice of G is motivated by the requirement to include the mass splitting [11], while the choice of G' — by the mass degeneracy.

In the case given by relations (3) and (4) the group G^* is chosen as either $U(1)$ or $SU'(2)$ [7]. After discovering the mesons $\psi_1(3095)$ [8] and $\psi_2(3684)$ [9] it seems that the only reasonable choice is to take $G = SU(3) \otimes G^*$ and $G' = SU(4)$. Correspondingly, the group G^* is now chosen as either $SU'(3)$ [7], or $SO(3)$ [12], or S_3 [13], which will be commented on in Section 4.

Physically, one should also motivate the choice of structure groups by the form of the Hamiltonian (2.4) in Ref. [1], but from the mathematical point of view it is better to avoid at this stage the Hilbert space description, so in the next section we are going to find conditions for the mappings v_e and v_n, specified below, yielded by the choice of the structure groups. Actually these mappings induce the mappings $V = V_e$ and $V = V_n$ between the corresponding Hilbert spaces, where V is defined as that in formula (2.2) of Ref. [1] and one could specify them by assuming that the corresponding state equation is Dirac-like, i.e. has the same form for all possible elementary systems.

In our framework we assume that $v_e : \mathbb{N}_e \to \mathbb{M}$ and $v_n : \mathbb{N}_n \to \mathbb{M}$ are some C^2-smooth mappings responsible for virtual particles such as π-mesons, arising from nuclear reactions such as $\mathrm{N} + \pi \rightleftarrows \mathrm{N}$. We specify N requiring that:

(i) the nuclear reaction in question evolves during an infinitesimal lapse of time $\Delta t = [t; t + \Delta t] \ll 1$, $\Delta t \cdot \Delta E \sim \hbar$, inside the hypertube in $\mathbb{M}$ corresponding to this lapse [14];

(ii) this reaction corresponds, during $\Delta t \neq 0$, to the nuclear field;

(iii) v_e complies with the interaction intensity $e_1{}^2/\hbar c$, where e_1 is the electric charge given by the formula (4.1) in Ref. [1] or, equivalently, by synopsis A in Ref. [3], the first paper;

(iv) v_n complies with the interaction intensity $g_1{}^2/\hbar c$, where g_1 is the nuclear charge given by the formula (5.1) in Ref. [1], or, equivalently, by synopsis A in Ref. [3], the first paper.

We are going to comment on the formulae for e and g. Let $\tilde{\mathbb{E}}_e$ be the standard four-dimensional manifold with $\operatorname{supp} \tilde{\mathbb{E}}_e = \operatorname{supp} \mathbb{M} \cup \operatorname{supp} \mathbb{N}_e$, endowed with the Minkowski metric. We further specify $\tilde{\mathbb{E}}_e$ by taking $\operatorname{supp} \tilde{\mathbb{E}}_e$ in the form of Cartesian product of the real line and either a k-pierced three-sphere or a three-torus. By a k-pierced sphere we understand the sphere with k pairs of nonoverlapping polar caps excluded and the corresponding points of the resulting boundaries of each pair of antipodal caps identified. Finally, we drill in the resulting manifolds also the additional nonintersecting holes corresponding to the particles in question [1], so we have to consider the submanifold $\tilde{\mathbb{E}}$ of $\tilde{\mathbb{E}}_e$ with $\operatorname{supp} \tilde{\mathbb{E}}_e = \operatorname{supp} \mathbb{M}$.

The formula for e mentioned above gives a sufficient condition for the existence of a C^1-differentiable mapping (*not* a diffeomorphism as quoted in Ref. [1] after Ref. [15]) $\Phi : \mathbb{E} \to \mathbb{M}$ such that the mapping Φ^* induced on singular 2-cycles on $\mathbb{E}$ gives rise to this formula [15]. By the proof given in Ref. [15] any C^1-differentiable mapping $\Phi : \mathbb{E} \to \mathbb{M}$ gives rise to such a formula. A necessary condition is given in Ref. [3], the first paper, at the end of Sect. 3, by the curvature properties of a principal toral bundle which may be expressed by the first Chern class of this bundle. Analogous statements are valid for $\mathbb{N}_n$.

3. Classification Scheme in Terms of Curvature Properties

We are going to establish relations between the curvature form Ω^2, given either by relations (1), (2) and (3) or (1′), (2′) and (3′), and the curved spacetime $\mathbb{M}$ via the C^2-smooth mappings v_e and v_n. Let us consider the system of mappings:

$$
\begin{array}{ccccc}
P(\mathbb{N}_e, G') & \underset{\pi^e}{\overset{s^e}{\rightleftarrows}} & \mathbb{N}_e & & \\
 & & \uparrow \mathrm{pr}^e & \searrow^{v_e} & \\
P(\mathbb{L}, G) & \underset{\pi}{\overset{s}{\rightleftarrows}} & \mathbb{L} & & \mathbb{M} \underset{\pi_N}{\overset{s_N}{\rightleftarrows}} P(\mathbb{M}, G) \\
 & & \downarrow \mathrm{pr}^n & \nearrow_{v_n} & \\
P(\mathbb{N}_n, G') & \underset{\pi^n}{\overset{s^n}{\rightleftarrows}} & \mathbb{N}_n & &
\end{array}
\tag{8}
$$

$$
\begin{array}{ccc}
 & F^p(\mathbb{N}_e) & \\
 & \mathrm{pr}_e{}^* = (\mathrm{pr}^e)^* \swarrow \quad \nwarrow v_e{}^* & \\
F^p(P(\mathbb{L}, G)) \underset{\pi^*}{\overset{s^*}{\rightleftarrows}} F^p(\mathbb{L}) & & F^p(\mathbb{M}) \underset{\pi_N{}^*}{\overset{s_N{}^*}{\rightleftarrows}} F^p(P(\mathbb{M}, G)), \\
 & \mathrm{pr}_n{}^* = (\mathrm{pr}^n)^* \nwarrow \quad \swarrow v_n{}^* & \\
 & F^p(\mathbb{N}_n) &
\end{array}
\tag{9}
$$

where π, π^e, π^n, π_N are the canonical projections, s, s^e, s^n, s_N are the corresponding cross sections, and π^*, s^*, etc. are the usual pull-back transformations of the corresponding vector spaces of all differential p-forms. Finally, let $\Omega_e{}^2$, $\Omega_n{}^2$, and $\Omega_N{}^2$ be the curvature forms on $P(\mathbb{N}_e, G')$, $P(\mathbb{N}_n, G')$, and $P(\mathbb{M}, G)$, respectively. Then, by the diagrams (8) and (9), we can relate the complex curvature form Ω^2 to the proton and neutron fields ($p = 2$):

$$s^* \Omega^2 = (s^{\mathrm{e}} \mathrm{pr}^{\mathrm{e}})^* \Omega_{\mathrm{e}}^2, \quad s^* \Omega^2 = (s^{\mathrm{n}} \mathrm{pr}^{\mathrm{n}})^* \Omega_{\mathrm{n}}^2,$$
$$\Omega_{\mathrm{e}}^2 = (s_{\mathrm{N}} v_{\mathrm{e}} \pi^{\mathrm{e}})^* \Omega_{\mathrm{N}}^2, \quad \Omega_{\mathrm{n}}^2 = (s_{\mathrm{N}} v_{\mathrm{n}} \pi^{\mathrm{n}})^* \check{\Omega}_{\mathrm{N}}^2,$$

where $\check{\Omega}_{\mathrm{N}}^2$ is a tensorial two-form of type ad G whose physical interpretation will be described below. Therefore

$$s^* \Omega^2 = \mathrm{pr_e}^* v_{\mathrm{e}}^* s_{\mathrm{N}}^* \Omega_{\mathrm{N}}^2, \qquad s^* \Omega^2 = \mathrm{pr_n}^* v_{\mathrm{n}}^* s_{\mathrm{N}}^* \check{\Omega}_{\mathrm{N}}^2. \tag{10}$$

Suppose that there is a connection Γ in $P(\mathbb{M}, G)$ which corresponds to the electromagnetic field in the usual sense[16]. Namely, suppose that Γ is characterized by a connection form ω_{e}, while let $\tilde{\omega}_{\mathrm{e}}$ be the two-form on $\mathbb{M}$ associated with the electromagnetic field, i.e. in local coordinates we have $\tilde{\omega}_{\mathrm{e}} = \frac{1}{2} F_{jk} \, \mathrm{d}x^j \wedge \mathrm{d}x^k$, where F_{jk} represents the electromagnetic field four-tensor. Consider the canonical Lie group representation homomorphism $\sigma_{\mathrm{e}}: G \to G^*$ and the corresponding induced Lie algebra homomorphism σ_{e}'. Then[16]

$$s_{\mathrm{N}}^* (\sigma_{\mathrm{e}}' \circ \Omega_{\mathrm{N}}^2) = i \, \tilde{\omega}_{\mathrm{e}} \tag{11}$$

and thus the next step is to determine the connection form ω_{e}.

The form ω_{e} satisfies the structure equation[17]

$$\mathrm{d}\omega_{\mathrm{e}}(\mathfrak{v}, \mathfrak{w}) = -\tfrac{1}{2} [\omega_{\mathrm{e}}(\mathfrak{v}), \omega_{\mathrm{e}}(\mathfrak{w})] + \Omega_{\mathrm{N}}^2(\mathfrak{v}, \mathfrak{w}), \tag{12}$$

where $\mathfrak{v}, \mathfrak{w} \in T_p P(\mathbb{M}, G)$, $p \in P(\mathbb{M}, G)$, and $[,]$ denotes the Lie bracket. Clearly, Eq. (12) expresses the relationship between the electromagnetic field and its four-vector potential. Equation (12) remains valid if $\mathfrak{v}$ and $\mathfrak{w}$ denote vector fields on $P(\mathbb{M}, G)$. If, in particular, $\mathfrak{v}$ and $\mathfrak{w}$ are horizontal, then[17]

$$\omega_{\mathrm{e}}([\mathfrak{v}, \mathfrak{w}]) = -2 \, \Omega_{\mathrm{N}}^2(\mathfrak{v}, \mathfrak{w}). \tag{13}$$

The Bianchi identity $\mathrm{D}\Omega_{\mathrm{N}}^2 = 0$, i.e. $\mathrm{d}\Omega_{\mathrm{N}}^2(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = 0$, where it suffices to consider horizontal vectors $\mathfrak{u}, \mathfrak{v}, \mathfrak{w} \in T_p P(\mathbb{M}, G)$, $p \in P(\mathbb{M}, G)$, is equivalent to the first four of the Maxwell equations.

Next, in analogy to Eq. (13), we define the form ω_{n}. To this end, let

$$\omega_{\mathrm{n}}([\mathfrak{v}, \mathfrak{w}]) = -2 \, \check{\Omega}_{\mathrm{N}}^2(\mathfrak{v}, \mathfrak{w}), \tag{14}$$

where $\mathfrak{v}$ and $\mathfrak{w}$ denote horizontal vector fields on $P(\mathbb{M}, G)$. More generally, we set

$$\mathrm{d}\omega_{\mathrm{n}}(\mathfrak{v}, \mathfrak{w}) = -\tfrac{1}{2} [\omega_{\mathrm{n}}(\mathfrak{v}), \omega_{\mathrm{n}}(\mathfrak{w})] + \check{\Omega}_{\mathrm{N}}^2(\mathfrak{v}, \mathfrak{w}), \tag{15}$$

where $\mathfrak{v}, \mathfrak{w} \in T_p P(\mathbb{M}, G)$, $p \in P(\mathbb{M}, G)$. Finally, let $\tilde{\omega}_{\mathrm{n}}$ be the two-form on $\mathbb{M}$ associated with the nuclear field, i.e. in local coordinates we have $\tilde{\omega}_{\mathrm{n}} = \frac{1}{2} G_{jk} \, \mathrm{d}x^j \wedge \mathrm{d}x^k$, where G_{jk} represents the nuclear field four-pseudotensor. Then, in analogy to Eq. (11), we define the Lie algebra homomorphism σ_{n}' by the relation

$$s_{\mathrm{N}}^* (\sigma_{\mathrm{n}}' \circ \check{\Omega}_{\mathrm{N}}^2) = i \, \tilde{\omega}_{\mathrm{n}}. \tag{16}$$

Finally, also from the physical point of view it is necessary to remark that formulae (1)—(16) give a theoretical possibility to classify elementary particles according to the adjoint representation ad in the Lie algebra $\mathfrak{g}$ of G. In fact, let $a \in G$ and let R_a denote the right translation by a, i.e. $R_a p = pa$ for $p \in P(\mathbb{M}, G)$. Then[17]

$$R_a^* \omega_{\mathrm{e}} = (\mathrm{ad}\, a^{-1}) \, \omega_{\mathrm{e}}, \qquad R_a^* \omega_{\mathrm{n}} = (\mathrm{ad}\, a^{-1}) \, \omega_{\mathrm{n}},$$

i.e.

$$\omega_{\mathrm{e}}((R_a)_* \mathfrak{v}) = (\mathrm{ad}\, a^{-1}) \, \omega_{\mathrm{e}}(\mathfrak{v}),$$
$$\omega_{\mathrm{n}}((R_a)_* \mathfrak{v}) = (\mathrm{ad}\, a^{-1}) \, \omega_{\mathrm{n}}(\mathfrak{v})$$

for every vector field $\mathfrak{v}$ on $P(\mathbb{M}, G)$ with the usual meaning of $(\ )_*$[17]. In this way, because of the definition (1), (3), (4) or (1), (5) of the curvature form Ω^2, mesons and baryons are accomodated in a single classification table which would improve the conventional approach if the necessary calculations were not too cumbersome.

4. The Choice of the Structure Groups

The choice of the structure group $G = SU(2) \otimes G^*$, $G^* = U(1)$ or $SU'(2)$, or of the structure group $G = SU(3) \otimes G^*$, $G^* = SU'(3)$ or $SO(3)$ or S_3, in the principal fibre bundle $P(\mathbb{L}, G)$, and thus also in $P(\mathbb{M}, G)$ as well as of $G' = SU(3)$ or $SU(4)$ in $P(\mathbb{N}_{\mathrm{e}}, G')$ and $P(\mathbb{N}_{\mathrm{n}}, G')$, may be now motivated by the following well known theorem[17]: Given an almost complex manifold $\mathbb{L}$ of real dimension $2n$, there is a natural one-to-one correspondence between any two of the following three sets: the hermitian metrics on $\mathbb{L}$, the reductions of the structure group of $C(\mathbb{L})$ to $U(n)$, and the cross sections of the associated bundle $C(\mathbb{L})/U(n)$ over $\mathbb{L}$, where $C(\mathbb{L})$ denotes the bundle of complex linear frames of $\mathbb{L}$ of real dimension $2n$ and

$$U(n) = O(2n) \cap SL(n; \mathbb{C}) \subset SU(n).$$

Thus, in particular, the optimal choice of the group G^* is closely connected with the effective determination of the hermitian metric of $\mathbb{L}$. The problem is theoretically solved, but at the moment effective procedures seem quite cumbersome. Therefore it is natural to recall the physical motivations for the particular choices of G^* mentioned before in the case given by relations (5): $G^* = SU'(3)$, $SO(3)$ or S_3.

The first as well as the third case is certainly richer in freedom (9 parameters) than the second one (6 parameters), so it is safer since recent experiments[8, 9] show that the process of discovering new particles is still in progress. Besides, $G^* = S_3$, the symmetric group of three objects, does not give double charges for mesons.

5. Structure of Particles

In the light of our considerations we can conclude that the particle in question is observed in the manifold $\mathbb{M}$ with metric g determined on the basis of the symmetry properties of this particle. In other words we can speak about a deformation of the space of observations introduced by the appearance of the particle. Thus we have a similar situation to the case where a particle produces the gravitational field, but now the origin of the potential field is not only connected with the mass, but also with other properties determined by the symmetry rules.

As it is known the equation of motion for the particle in the manifold $\mathbb{M}$ with metric g can be written in the form following the derivation of Dirac's equation in arbitrary coordinate systems. A generalization of pseudo-riemannian structures to include spinor variables was developed by Tetrode[18] and by Fock and Ivanenko[19]. In the notation of Schmutzer[20], the general covariant form of Dirac's equation is (cf. also Ref. [21]):

$$\gamma^k [(\partial/\partial x^k) + \Gamma_k]\, \psi + (m c/\hbar)\, \psi = 0\,, \qquad (17)$$

where m denotes the rest mass of the particle in question and γ^k are the Dirac matrices obtained, by the commutation rules

$$\gamma^j \gamma^k + \gamma^k \gamma^j = 2\, g^{jk}$$

and the Γ_k are the spinor connections (i.e. the generalized Christoffel symbols) determined by

$$\Gamma_k = \tfrac{1}{4}\, \gamma^j (\gamma_{j|k} - \{{}_j{}^l{}_k\}\, \gamma_l) - \tfrac{1}{32}\, \mathrm{Tr}(\gamma\, \gamma^j \gamma_{j|k})\, \gamma$$

with

$$\gamma = \tfrac{1}{24}\, \varepsilon_{jklm}\, \gamma^j \gamma^k \gamma^l \gamma^m\,,$$

where ε_{jklm} denotes the totally antisymmetric Levi-Civita tensor.

Thus, as we can see, the equation of motion (17) contains in general additional terms equivalent to the interaction of this particle with the external field. Therefore, we have also in our case a situation in which the free particle is described by the equation of motion containing additional forces following from the existence of this particle.

From the physical point of view a free particle should be described in terms of the constant Minkowski metric g^0 since the equation of motion given in this metric leads to the motion of a free particle, i.e. to the motion with the constant velocity. Thus, the observation related to the curved Minkowski space-time suggests that the motion of the particle described by the metric g is of the character of a relative motion with respect to the original position of the particle in which the structural field is produced. This picture can be interpreted as the consideration of a system with the internal structure.

We assume that the motion of the free particle corresponds to the change of the curved Minkowski space-time and for this reason the position of the particle can be represented by the centre of mass of the system containing the particle observed in the space of observations $\mathbb{M}$ and the original position of the particle with respect to this space. The fictitious force occuring in the curved Minkowski space-time represents an interaction between two components of the particle.

More precisely speaking, we can treat the particle as a composition of two particles interacting with each other. The motion of the centre of mass of this composition is observed as the behaviour of the particle while the relative motion can be related to the structural properties of the particle. The relative motion appears for the particle with the reduced mass μ which is determined by the rest mass m and the energy E of the relative motion considered in the space of observations $\mathbb{M}$, where $\mu c^2 = E + m c^2$. We would like to stress here that the energy E is negative for the bound particles, so $\mu < m$.

In the case where we observe two particles in the manifold $\mathbb{M}$, its metric g results from the composition of two interacting fields produced by these particles. If we denote the distance between them by R, we can obtain the effective metric g as a function of R. Then the motion of the particle in question can be treated as a motion in an external field produced by the source localized at a point of the space $\mathbb{M}$ situated at the distance R from this particle. The character of the interaction is determined by the form of the metric tensor $g(R)$ following from the symmetry properties of the particles.

If we accept the point of view presented by Sakharov[5] that the metric g is given, in local coordinates, by the formula (2), i.e. it is determined by the field four-vector ψ which is a solution of Eq. (17), we obtain a nonlinear differential equation for ψ. In this way the metric g is determined.

As an example we may consider the metric with elements $g^{jk} = M_j \delta^{jk}$, where $M_1 = M_2 = M_3 = M$ and $M = -1$, M being the constant corresponding to the value of the magnetic moment counted in the units of nuclear Bohr magnetons. In this case $\Gamma_k = 0$, whereas the matrices γ^k can be expressed as linear combinations of the Dirac matrices $\gamma_D{}^j$, namely

$$\gamma^k = a_j{}^k \gamma_D{}^j . \tag{18}$$

Then the coefficients $a_j{}^k$ satisfy the relations

$$\sum_j a_j{}^k a_j{}^l = g^{kl} \tag{19}$$

so that $a_j{}^k = \sqrt{M_k}\, \delta_j{}^k$ (not summed). Considering the transformation of the operator of momentum given by $\partial/\partial x^j = a_j{}^k \partial/\partial x^k$ we conclude that Eq. (17) attains the form $[M_k \gamma_D{}^k \partial/\partial x^k + (m c/\hbar)] \psi = 0$ already discussed in Ref. [1] as corresponding to the Hamiltonian (7.4). Thus we arrive at the conclusion that a particle with a given magnetic moment M generates the manifold with metric $g^{jk} = M \delta^{jk}$, since then we can obtain a description of the properties of the particle yielding the correct magnetic moment.

In the case of an arbitrary metric g^{jk} we may proceed in the same way, namely, the matrices γ^k can be determined as a linear combination (18) whose coefficients are given by Equation (19). Then, in dependence of a chosen metric g^{jk}, Dirac's equation (17) attains the form yielded by the relations (18) and the transformation for Γ_k which follows from the transformation for $\gamma_D{}^j$.

6. Physical Comments

Before we make the final physical comments we should have in mind the existing models of elementary particles even if we act independently of them. As a general reference we quote five recent survey articles[22].

We are now in a position to describe the physical meaning of the tensorial two-form $\breve{\Omega}_N{}^2$ defined in Sect. 3: it may be regarded as a fictitious curvature form corresponding to the nuclear field. Besides we conclude that, since the curvature form Ω^2 is already known (cf. Introduction) as well as the projections pr_e, pr_n and the cross sections s_e, s_n corresponding to the canonical projections π^e, π^n, respectively, it remains to determine the two-forms $\tilde{\omega}_e$ and $\tilde{\omega}_n$. In principle they are also known if the electromagnetic and nuclear fields are known, but in fact we are dealing with the quantized electric charges, magnetic charges, nuclear charges, charges of nuclear monopoles, and masses[1]. We should like to stress that the existence of nuclear monopoles is even more probable than the appearance of magnetic monopoles[23] since the nuclear monopoles can be motivated in terms of the concept of a quark considered as a particle of weak interaction.

In view of the fact that some bound states of elementary particles, e.g. e^+e^-, may be described by the Dirac equations for two particles with the spin $\frac{1}{2}$ (cf. Ref. [24]), the generalization of our considerations relies upon taking into account the metric of the manifold generated by two particles simultaneously and described, in the case of Sakharov's metric[5], by the wave vector ψ with eight components. Then these components can be determined with help of the equation of Królikowski and Rzewuski[24], considered on the manifold endowed with Sakharov's metric.

There are some nonstandard approaches. We have already quoted the papers of Sakharov[5] and von Westenholz[3] as concerned with manifolds endowed with suitable metrics, but there is even a more nonstandard approach due to Grauert[25] who suggests the use of a geometrical tool much simpler than the pseudo-riemannian structure: it consists just of a set S of line segments in space-time, briefly speaking. In terms of this new structure he was able to obtain the not observable magnitude of vacuum polarization, to define the production of virtual photons, and to evaluate the correct value of the elementary charge.

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